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SMALL x, SATURATION AND THE HIGH ENERGY LIMIT OF QCD

Edmond IANCU
CERN, Theory Division
CH-1211 Geneva 23, Switzerland

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1 The Colored Glass Condensate

At very small values of the Bjorken x-variable, one expects QCD to enter a new regime which is caracterized by parton saturation and very high values of the QCD field strength $F^a_{\mu\nu} \sim 1/g$. Saturation, which is a limitation on the maximum phase-space parton density that can be reached in the hadron wavefunction, may have been already observed at HERA [2], and should play an important role in the very early stages of relativistic heavy ion collisions at RHIC and LHC [3] (and Refs. therein). In the saturation regime, the individual parton-parton interactions may be weak [b] (which we shall assume in what follows; i.e., we assume that $g \ll 1$), but the parton densities are so large that the system becomes strongly non-perturbative. Thus, at a theoretical level, understanding saturation is a challenging and fascinating problem where one has to deal with fully non-linear QCD. This is reminiscent of a similar problem in high temperature QCD where perturbation theory breaks down at the soft scale g^2T because of the large thermal occupation numbers of the soft gluons [4] (and Refs. therein).

The efforts toward understanding the region of high gluon density have uncovered a new form of matter which is formed from these gluons [1]. This matter is universal in that it should describe the high gluon density part of any hadron and nuclear wavefunction at small x. The combination of high density and the fact that gluons are massless bosons leads naturally to the expectation that this matter is a Bose condensate. Since the gluons carry color and local color is a gauge dependent quantity, any gauge invariant formulation will neccessarily involve an average over all colors to restore the invariance. This averaging procedure bears a formal ressemblance to the averaging over background fields done for spin glasses [5]. The matter is therefore called the Colored Glass Condensate (CGC).

It should be emphasized that this picture holds, strictly speaking, only in the infinite-momentum frame, where the hadron propagates almost at the speed of light, and thus appears as an infinitesimally thin two-dimensional sheet (by Lorentz contraction). In this frame, the

[a] Work done in collaboration with Andrei Leonidov and Larry McLerran [1].

[b] This is the case if the saturation momentum Q_s (cf. eq. (11) below) is large enough; e.g., at LHC, one expects $Q_s \sim 2 - 3$ GeV.

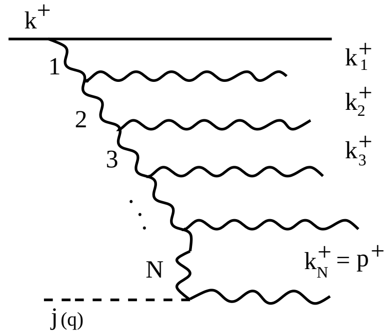

Figure 1: A parton cascade leading to a small-x virtual gluon.

parton interpretation makes sense and deep inelastic scattering (DIS) proceeds via the instantaneous absorbtion of the external probe (e.g., a virtual photon γ^* with 4-momentum q^μ) by some parton in the hadron. The Bjorken x_B parameter is defined as $x_B \equiv Q^2/2P \cdot q$, where $Q^2 \equiv -q^\mu q_\mu$, and $P^\mu = \delta^{\mu +} P^+$, with large P^+, is the hadron 4-momentum [c]. By kinematics, x_B coincides with the longitudinal momentum fraction $x \equiv p^+/P^+$ of the struck parton: $x_B = x$.

At $x \ll 1$, the gluon density increases faster, and is the driving force toward saturation [d]. The dynamics that leads to this increase is the quantum evolution toward small-x : in a parton cascade initiated by some *fast* parton (i.e., a hadron constituent with a relatively large longitudinal momentum $k^+ \sim P^+$, e.g., a valence quark), the number of radiated gluons increases exponentially with the rapidity gap $\Delta\tau \equiv \ln(k^+/p^+) \sim \ln(1/x)$ between the original parton and the *soft* (i.e., small p^+: $p^+ = xP^+ \ll P^+$) final gluon which is struck by the external current (see Fig. 1). This (BFKL) picture—which assumes the radiated gluons to behave as free partons—ceases to be valid at very small x, where the gluon density is so large that the radiated gluons overlap each other in the transverse plane. This is the onset of saturation.

This is also the regime where the description in terms of a colored class condensate becomes appropriate: Corresponding to the strong ordering in longitudinal momenta along the cascade,

$$k^+ \equiv k_0^+ \gg k_1^+ \gg k_2^+ \gg \cdots \gg k_N^+ \equiv p^+ \,, \tag{1}$$

there is a similar hierarchy among the lifetimes of the radiated gluons (since the latter are proportional to the respective k^+ momenta): Softer a gluon is, shorter is its lifetime.

This hierarchy in time stays at the basis of a *quenched approximation* [6,7,8,9] for the "fast" degrees of freedom: when "seen" by the soft modes with very short lifetimes, the modes with larger p^+ appear as frozen (no dynamics) and can be replaced by a *static* (i.e., independent of x^+) and *random* color charge configuration, with density $\rho_a(x^-, x_\perp)$. (This is random since the soft gluons can belong to different cascades.) The spatial correlations of the effective charge $\rho_a(x^-, x_\perp)$ reflect the quantum dynamics at large longitudinal momenta, and are encoded in a statistical weight function $W[\rho]$. Because of the Lorentz contraction, we can write $\rho_a(x^-, x_\perp) \approx \delta(x^-)\rho_a(x_\perp)$, and W is a functional of the *superficial* charge density $\rho_a(x_\perp)$ alone.

Thus, in order to compute soft correlations in this approximation, one has to first study the (quantum) dynamics of the soft gluons in the presence of a given color charge ρ_a, and then perform a (classical) average over ρ_a, with weight function $W[\rho]$. Clearly, the latter will depend upon what we call "soft" and "fast", i.e., upon the separation scale Λ between *fast* ($p^+ > \Lambda$) and *soft* ($p^+ < \Lambda$) degrees of freedom. We thus write $W[\rho] \equiv W_\Lambda[\rho]$.

[c] We use light-cone vector notations: for some arbitrary vector p^μ, we write $p^\mu = (p^+, p^-, \mathbf{p}_\perp)$, with $p^+ \equiv (1/\sqrt{2})(p^0 + p^3)$, $p^- \equiv (1/\sqrt{2})(p^0 - p^3)$, and $\mathbf{p}_\perp \equiv (p^1, p^2)$.

[d] To directly measure the gluon density, it is convenient to consider a Gedanken experiment where the DIS is initiated by the "current" $j \equiv -\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a$ which couples directly to gluons.

For instance, the 2-point correlation function is obtained as (in the light-cone gauge $A^+ = 0$)

$$\langle \mathrm{T}\, A^\mu(x) A^\nu(y) \rangle \,=\, \int \mathcal{D}\rho\; W_\Lambda[\rho] \left\{ \frac{\int^\Lambda \mathcal{D}A\;\, A^\mu(x) A^\nu(y)\; \mathrm{e}^{\,iS[A,\rho]}}{\int^\Lambda \mathcal{D}A\;\, \mathrm{e}^{\,iS[A,\rho]}} \right\}, \tag{2}$$

with the functional integral running only over soft gluon fields $A_a^\mu(p)$ with $p^+ < \Lambda$, and the action $S[A,\rho]$ describing the dynamics of these fields in the presence of the classical source ρ_a, in the eikonal appoximation [8]: $S = S_{YM} + S_W$, where $S_{YM} = \int d^4x(-F_{\mu\nu}^2/4)$ is the usual Yang-Mills action, and (with $\tilde{x} \equiv (x^-, x_\perp)$, and T denoting time-ordering of color matrices):

$$S_W \,\equiv\, \frac{i}{N_c} \int d^3\tilde{x}\, \mathrm{Tr}\, \left\{ \rho(\tilde{x})\, \mathrm{T}\, \exp\left[ig \int dx^+ A^-(x^+, \tilde{x}) \right] \right\}. \tag{3}$$

The double averaging in eq. (2) is very similar to the one performed for spin systems in a random external magnetic field [5] (which plays the role of ρ in the above equation), and supports the physical picture of the saturation regime as a coloured glass condensate.

Of course, the final results for soft correlators must be independent of the arbitrary separation scale Λ. The Λ-dependence of the weight function $W_\Lambda[\rho]$ must cancel against the cutoff dependence of the quantum theory for the soft modes. This constraint can be formulated as a renormalization group equation for $W_\Lambda[\rho]$ [9], to be presented in Sect. 3 below.

2 Saturation in the classical approximation

Consider first the simple approximation where the quantum path integral in eq. (2) is evaluated in the saddle-point (or classical) approximation $\delta S/\delta A^\mu = 0$, and the weight function is taken simply as a Gaussian (this is the McLerran-Venugopalan model [6], originally formulated for a large nucleus for which the Gaussian approximation is expected to work better) :

$$W_\Lambda[\rho] \simeq \exp\left\{ -\frac{1}{2\mu_\Lambda^2} \int d^2x_\perp\, \rho_a^2(x_\perp) \right\}. \tag{4}$$

Here, μ_Λ^2 is to the total color charge squared (per unit area) of the partons with $p^+ > \Lambda$.

The classical approximation of eq. (2) reads (with $\tilde{x} \equiv (x^-, x_\perp)$) :

$$\langle A^\mu(x^+, \tilde{x}) A^\nu(x^+, \tilde{y}) \rangle_{cl} \,=\, \int \mathcal{D}\rho\; W_\Lambda[\rho]\, \mathcal{A}^\mu(\tilde{x}) \mathcal{A}^\nu(\tilde{y})\,, \tag{5}$$

where $\mathcal{A}^\mu(\tilde{x})$ is the solution to the classical Yang-Mills equations with source $\rho_a(\tilde{x})$,

$$[D_\nu, F^{\nu\mu}] \,=\, g\delta^{\mu +}\delta(x^-)\rho_a(x_\perp)\,, \tag{6}$$

and is time-independent, like ρ_a itself. The LC gauge solution can be written as [1]:

$$\mathcal{A}^+ = \mathcal{A}^- = 0, \qquad \mathcal{A}^i(\tilde{x}) = \theta(x^-)\mathcal{A}_+^i(x_\perp) + \theta(-x^-)\mathcal{A}_-^i(x_\perp), \tag{7}$$

with $\mathcal{A}_+^i(x_\perp)$ and $\mathcal{A}_-^i(x_\perp)$ related to $\rho(x_\perp)$ via a non-linear equation. Note that the vector potential $\mathcal{A}^i(x^-, x_\perp)$ is discontinuous at $x^- = 0$, so the associated electric field $\mathcal{F}^{+i} \equiv \partial^+ \mathcal{A}^i$ is localized at the light-cone (i.e., within the support of the source):

$$\mathcal{F}^{+i}(\tilde{x}) = \delta(x^-)\Big(\mathcal{A}_+^i(x_\perp) - \mathcal{A}_-^i(x_\perp) \Big). \tag{8}$$

By using this approximation, let us compute the gluon distribution function, that is, the number of gluons per unit of x in the hadron wavefunction having transverse momentum less than Q. This is defined as (with $\tilde{k}^\mu \equiv (k^+, \mathbf{k}_\perp)$ and $k^+ = xP^+$) :

$$xG(x, Q^2) \,=\, \frac{1}{\pi} \int \frac{d^2k_\perp}{(2\pi)^2}\, \Theta(Q^2 - k_\perp^2) \left\langle F_a^{+i}(x^+, \tilde{k}) F_a^{+i}(x^+, -\tilde{k}) \right\rangle. \tag{9}$$

In the classical approximation (8), $F^{i+}(x^+,\tilde{k}) \approx \mathcal{F}^{i+}(k_\perp)$ is independent of both x^+ and k^+, and (with $\Delta\mathcal{A}^i \equiv \mathcal{A}^i_+ - \mathcal{A}^i_-$, and the classical average defined as in eq. (5)) :

$$xG_{cl}(x,Q^2) \;=\; \frac{A}{\pi}\int^{Q^2} \frac{d^2k_\perp}{(2\pi)^2}\int d^2x_\perp\, \mathrm{e}^{-ik_\perp\cdot x_\perp}\Big\langle \Delta\mathcal{A}^i_a(0_\perp)\,\Delta\mathcal{A}^i_a(x_\perp)\Big\rangle_{cl}. \tag{10}$$

Here, A is the hadron transverse aria, and we have assumed transverse homogeneity. Thus, $xG_{cl}(x,Q^2)$ is independent of x, which reflects the absence of quantum evolution in the present, classical approximation. With the Gaussian weight function (4), and the non-linear classical solution (7), the classical gluon distribution (10) can be computed exactly [7,10]. One obtains:

$$\Big\langle \Delta\mathcal{A}^i_a(0_\perp)\,\Delta\mathcal{A}^i_a(x_\perp)\Big\rangle_{cl} \;=\; \frac{N_c^2-1}{\pi\alpha_s N_c}\,\frac{1-\mathrm{e}^{-x_\perp^2\ln(x_\perp^2\Lambda_{QCD}^2)Q_s^2/4}}{x_\perp^2}\,, \tag{11}$$

where $Q_s \propto \alpha_s\mu_\Lambda$ is the *saturation momentum*, and is a priori a function of Λ. Remarkably, this equation displays saturation via non-linear effects in the classical solution. This interpretation can be made sharper by going to momentum space. If $N(k_\perp)$ is the Fourier transform of (11),

$$N(k_\perp) \,\propto\, \alpha_s(Q_s^2/k_\perp^2) \quad \text{for} \quad k_\perp^2 \gg Q_s^2, \tag{12}$$

which is the normal perturbative behaviour, but

$$N(k_\perp) \,\propto\, \frac{1}{\alpha_s}\,\ln\frac{k_\perp^2}{Q_s^2} \quad \text{for} \quad k_\perp^2 \ll Q_s^2, \tag{13}$$

which shows a much slower increase, i.e., saturation, at low momenta (with $k_\perp \gg \Lambda_{QCD}$ though). According to eq. (11), saturation is also a statement about the maximum field intensity that can be reached in the system: the classical field never becomes larger than $A^i \sim 1/g$. This is the maximal occupation number for a classical field, since larger occupation numbers are blocked by repulsive interactions of the gluon field.

3 The non-linear evolution equation

In writing down the effective theory for soft gluons in eq. (2), we have assumed that the influence of the fast gluons can be reproduced by a classical color source ρ with a peculiar structure (time-independent, and localized at $x^-=0$), and some (still unspecified) weight function $W_\Lambda[\rho]$. In this section, we show how to construct this effective theory, step by step, by integrating quantum fluctuations in successive layers of p^+.

To this aim, it is convenient to consider a sequence of two effective theories ("Theory I" and "Theory II") defined as in eq. (2), but with different separation scales: Λ in the case of Theory I, and $b\Lambda$ for Theory II, with $b \ll 1$. That is, Theory II differs from Theory I in that the "semi-fast" fields with longitudinal momenta between $b\Lambda$ and Λ have been integrated out, and the associated correlations have been incorporated at tree-level, within the new weight function $W_{b\Lambda}$. The difference $\Delta W \equiv W_{b\Lambda} - W_\Lambda$ can be obtained [1] by matching calculations of soft ($k^+ \ll b\Lambda$) gluon correlations in both theories. The result can be expressed as an evolution equation for $W_\tau[\rho]$ (with $\tau \equiv \ln(P^+/\Lambda)$) with respect to variations in τ.

The quantum corrections due to the semi-fast fields have to be computed to leading logarithmic accuracy (LLA), that is, to leading order in $\alpha_s\ln(1/b)$—indeed, it is only to this accuracy that the hierarchy of scales in eq. (1) is satisfied, and the matching is possible—, but to all orders in the classical fields and sources (since, in the saturation regime of interest, the non-linear effects are so strong that cannot be expanded in perturbation theory). This specifies the accuracy of the evolution equations to be obtained.

To this accuracy, there are only two contributions to ΔW : the one- and two-point correlators of the fluctuating colour charge $\delta\rho_a(x)$ of the semi-fast gluons. Specifically, one obtains [9,1]

$$\begin{aligned}\langle\delta\rho_a(x)\rangle_\rho &\approx \alpha_s \log(1/b)\,\delta(x^-)\,\sigma_a(x_\perp),\\ \langle\delta\rho_a(x)\delta\rho_b(y)\rangle_\rho &\approx \alpha_s \log(1/b)\,\delta(x^-)\,\chi_{ab}(x_\perp,y_\perp)\,\delta(y^-),\end{aligned} \tag{14}$$

while all the n-point correlators with $n \geq 3$ are of higher order in α_s. In these equations, $\langle\cdots\rangle_\rho$ denotes a quantum expectation value over the semi-fast fields at fixed ρ. Furthermore, $\sigma_a(x_\perp)$ and $\chi_{ab}(x_\perp, y_\perp)$ are generally non-linear functionals of $\rho(x_\perp)$ given by one-loop diagrams within Theory I (with loop momenta restricted to the strip: $b\Lambda < |p^+| < \Lambda$).

In terms of these functions, the evolution equation for $W_\tau[\rho]$ reads [e] [9]

$$\frac{\partial W_\tau[\rho]}{\partial\tau} = \alpha_s\left\{\frac{1}{2}\frac{\delta^2}{\delta\rho_x\delta\rho_y}[W_\tau\chi_{xy}] - \frac{\delta}{\delta\rho_x}[W_\tau\sigma_x]\right\}. \tag{15}$$

This functional equation is equivalent to an infinite hierarchy of ordinary equations for the correlators of the charge density. For instance, by multiplying eq. (15) with $\rho_x\rho_y$ and functionally integrating over ρ, one obtains an evolution equation for the two-point function:

$$\frac{d}{d\tau}\langle\rho_x\rho_y\rangle_\tau = \alpha_s\,\langle\chi_{xy} + \rho_x\sigma_y + \sigma_x\rho_y\rangle_\tau\,, \tag{16}$$

which in general, however, involves also the higher n-point functions, via σ and χ. But in the weak source approximation, i.e., with σ and χ computed to lowest order in ρ, this becomes a closed equation for $\langle\rho\rho\rangle$ which has been shown [8] to be equivalent to the BFKL equation, as necessary on physical grounds. This is a highly non-trivial check of the effective theory in eq. (2).

In order to study saturation, however, one needs eq. (15) in the regime of strong background fields and sources ($\mathcal{A}^i \sim 1/g$ and $\rho \sim 1/g^2$; cf. eqs. (11) and (6)), which requires for an *exact* calculation of the coefficients σ and χ. This has been done recently [1], via a lenthy calculation which had to cope with difficulties related to gauge-fixing, the axial poles in the gluon propagator, and the proper definition of the singular limit $\rho(x) \to \delta(x^-)\rho(x_\perp)$. This makes possible to look for solutions to eq. (15). Note that, formally, this is like a functional Schrödinger equation in imaginary "time" τ. An interesting possibility is that the "Hamiltonian" in its r.h.s. has an eigenfunction $\mathcal{W}[\rho]$ of maximum eigenvalue λ. This would lead to an universal behavior of $W_\tau[\rho]$ in the small-x, or high-energy, limit: $W_\tau[\rho] \longrightarrow_{\tau\to\infty} \mathrm{e}^{\tau\lambda}\,\mathcal{W}[\rho]$, with τ-independent $\mathcal{W}[\rho]$.

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[e] In condensed notations where, e.g., ρ_x stands for $\rho_a(x_\perp)$, and repeated indices are understood to be summed (integrated) over.